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Bipartite graphs with small third Laplacian eigenvalue[☆]

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Abstract

In this paper, all connected bipartite graphs are characterized whose third largest Laplacian eigenvalue is less than three. Moreover, the result is used to characterize all connected bipartite graphs with exactly two Laplacian eigenvalues not less than three, and all connected line graphs of bipartite graphs with the third eigenvalue of their adjacency matrices less than one.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E . Denote by $D(G) = \text{diag}(d_u, u \in V)$ (d_u is the degree of a vertex u) and $A(G)$ the degree diagonal and the adjacency matrices of G , respectively. Then $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G (see e.g., [10]). Clearly, $L(G)$ is a positive semidefinite matrix. So the eigenvalues of $L(G)$ can be denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G) = 0$ and are called the *Laplacian eigenvalues* of G . Throughout this paper, we always assume that G has at least four vertices.

Since the algebraic properties of the Laplacian matrix are used as a bridge between different kinds of structural properties of the graph, the relation between the structural (combinatorial, topological) properties of the graph and the algebraic ones of

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the corresponding Laplacian matrix is a very interesting topic. The reader may be referred to [6,10,11,15] and the references therein. In the work of determining graphs with small number of Laplacian eigenvalues exceeding a given value, Grone et al. [5], Merris [9] and Petrovic et al. [13] have made a lot of investigation. On the other hand, Gutman et al. [7,8] discovered some connections between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of the underlying molecular graphs. Hence the results obtained in this work can be of interest in the photoelectron spectroscopy of organic compounds. On the background of spectral graph theory and nonnegative matrix theory, the reader may be referred to [1,2,4].

This paper is organized as follows. In Section 2, we study some bipartite graphs with $\lambda_3(G) < 3$. These results are used, in Section 3, to determine all connected bipartite graphs with $\lambda_3(G) < 3$. In Section 4, we characterize all connected bipartite graphs with exactly two Laplacian eigenvalues not less than three. In the last section, all line graphs of bipartite graphs with the third eigenvalue of adjacency matrices less than 1 are characterized.

2. Some bipartite graphs with $\lambda_3(G) < 3$

The following is a well-known result on the Laplacian eigenvalues (see e.g., [5]). It will be often used in this paper.

Lemma 2.1. *Let G be a simple graph of order n . If H is a subgraph of G of order $m \leq n$ (not necessary an induced subgraph). Then for $i = 1, \dots, m$, we have*

$$\lambda_i(G) \geq \lambda_i(H). \quad (1)$$

We study the set \mathcal{G} of all connected graphs G with the property

$$\lambda_3(G) < 3. \quad (2)$$

Property (2) is hereditary by Lemma 2.1. A connected graph G is called a *forbidden subgraph* for \mathcal{G} if $G \notin \mathcal{G}$ but any proper subgraph is in \mathcal{G} . By a direct calculation, we have the following simple result:

Lemma 2.2. *The following graphs in Fig. 1 are forbidden subgraphs for \mathcal{G} , i.e., $\lambda_3(H_i) \geq 3$ for $i = 1, \dots, 8$.*

We are ready to introduce some graphs which satisfy (2).

Lemma 2.3. *Let $G(m, p, q, r, s)$ be the graph of order $n = 2p + q + 2r + s + m + 4$ in Fig. 2, where $m, p, q, r, s \geq 0$. Then $\lambda_3(G(m, p, q, r, s)) < 3$, i.e., $G(m, p, q, r, s) \in \mathcal{G}$.*

Proof. By Lemma 2.1, without loss of generality, we may assume that $m = p = r \geq 1$ and $q = s = 0$, since $G(m, p, q, r, s)$ can be regarded the connected subgraph of $G(m + p + q + r + s, m + p + q + r + s, 0, m + p + q + r + s, 0)$. By a direct calculation,

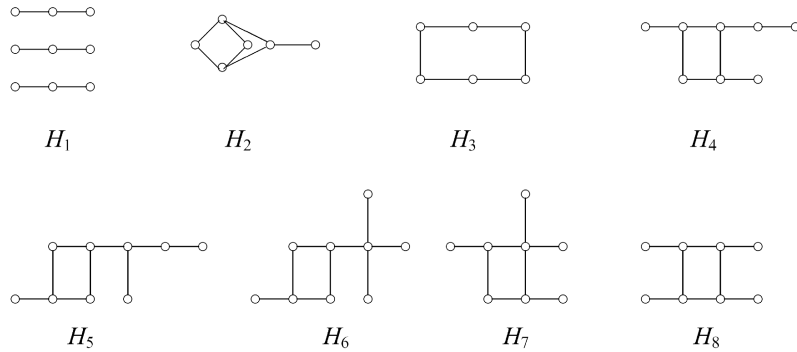


Fig. 1.

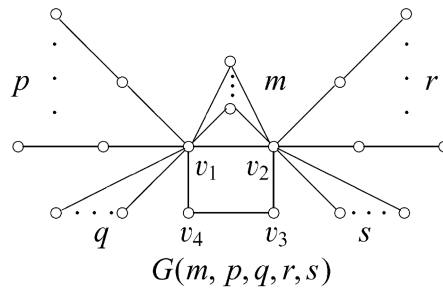


Fig. 2.

we can show that the characteristic polynomial of $L(G(p, p, 0, p, 0))$ is equal to

$$\begin{aligned} & \lambda(\lambda - 2)^{p-1}(\lambda^2 - 3\lambda + 1)^{2p-2} \\ & \times (\lambda^8 - (4p + 16)\lambda^7 + (4p^2 + 52p + 107)\lambda^6 - (40p^2 + 274p + 388)\lambda^5 \\ & + (155p^2 + 750p + 825)\lambda^4 - (292p^2 + 1132 + 1034)\lambda^3 \\ & + (274p^2 + 918p + 724)\lambda^2 - (116p^2 + 358p + 248)\lambda + (15p^2 + 52p + 32)) \\ & := \lambda(\lambda - 2)^{p-1}(\lambda^2 - 3\lambda + 1)^{2p-2}g(\lambda). \end{aligned}$$

Clearly, it is easy to see that $g(0) > 0$, $g(\frac{1}{2}) < 0$, $g(1) > 0$, $g(2) < 0$, $g(\frac{7}{3}) > 0$, $g(\frac{5}{2}) < 0$, $g(3) > 0$, $g(2p+3) < 0$ and $g(2p+4) > 0$. Hence $g(\lambda) = 0$ has exactly two roots not less than three (in the sequel, we always use this method to prove the distributions of the roots of the equations and omitted). Therefore $L(G(p, p, 0, p, 0))$ has exactly two eigenvalues not less than three. So $\lambda_3(G(m, p, q, r, s)) < 3$. \square

Let, $G_1(p, q, r, s) = G(0, p, q, r, s)$; $G_2(p, q, r, s)$ be the graph obtained from graph $G(0, p, q, r, s)$ by deleting the edge v_1v_2 ; $G_3(p, q, r, s)$ be the graph obtained from graph $G(0, p, q, r, s)$ by deleting the vertices v_3, v_4 and $G_4(m, p, q, r, s)$ be the graph

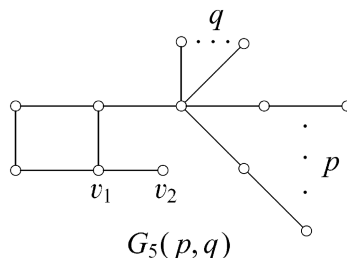


Fig. 3.

obtained from graph $G(m, p, q, r, s)$ by deleting the vertices v_3, v_4 and the edge $v_1 v_2$. From Lemma 2.3, we have the following result.

Corollary 2.4. For $i = 1, 2, 3$, $\lambda_3(G_i(p, q, r, s)) < 3$ and $\lambda_3(G_4(m, p, q, r, s)) < 3$.

Lemma 2.5. Let $G_5(p, q)$ be the graph of order $n = 2p + q + 6$ in Fig. 3, where $p \geq 0$, $q \geq 0$. Then $\lambda_3(G_5(p, q)) < 3$.

Proof. By Lemma 2.1, without loss of generality, we may assume that $q = 0$ and $p \geq 2$. By a direct calculation, we can show that the characteristic polynomial of $L(G_5(p, 0))$ is equal to

$$\begin{aligned} & \lambda(\lambda^2 - 3\lambda + 1)^{p-1}(\lambda^7 - (p + 15)\lambda^6 + (13p + 89)\lambda^5 - (64p + 268)\lambda^4 \\ & \quad + (150p + 435)\lambda^3 - (169p + 373)\lambda^2 + (78p + 155)\lambda - (8p + 24)) \\ & := \lambda(\lambda^2 - 3\lambda + 1)^{p-1}g(\lambda). \end{aligned}$$

It is easy to see that $g(0) < 0$, $g(\frac{1}{2}) > 0$, $g(1) < 0$, $g(2) > 0$, $g(\frac{5}{2}) < 0$, $g(3) > 0$, $g(5) < 0$, $g(p+4) > 0$. Then $g(\lambda) = 0$ has exactly two roots not less than three. Hence $L(G_5(p, 0))$ has exactly two eigenvalues not less than three. So $\lambda_3(G_5(p, q)) < 3$. \square

Let $G_6(p, q)$ be the graph obtained from $G_5(p, q)$ by deleting the vertex v_2 and the edge $v_1 v_2$.

Lemma 2.6. Let $G_7(p, q)$ be the graph of order $n = 2p + q + 6$ in Fig. 4, where $p \geq 0$, $q \geq 0$. Then $\lambda_3(G_7(p, q)) < 3$.

Proof. By Lemma 2.1, without loss of generality, we may assume that $q = 0$ and $p \geq 4$. By a direct calculation, we can show that the characteristic polynomial of $L(G_7(p, 0))$ is equal to

$$\begin{aligned} & \lambda(\lambda - 2)(\lambda^2 - 3\lambda + 1)^{p-1}(\lambda^6 - (p + 13)\lambda^5 + (11p + 62)\lambda^4 - (41p + 137)\lambda^3 \\ & \quad + (62p + 145)\lambda^2 - (34p + 70)\lambda + (4p + 12)) \\ & := \lambda(\lambda - 2)(\lambda^2 - 3\lambda + 1)^{p-1}g(\lambda). \end{aligned}$$

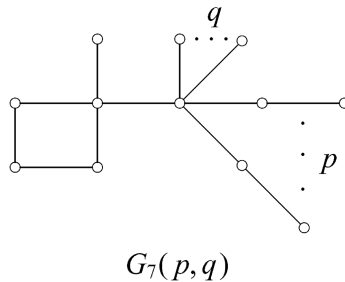
 $G_7(p, q)$

Fig. 4.

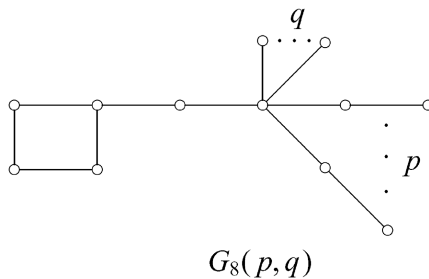
 $G_8(p, q)$

Fig. 5.

It is easy to see that $g(0) > 0$, $g(\frac{1}{2}) < 0$, $g(1) > 0$, $g(2) < 0$, $g(3) > 0$, $g(6) < 0$, $g(p+4) > 0$. Hence $g(\lambda) = 0$ has exactly two roots not less than three. So $L(G_7(p, 0))$ has exactly two eigenvalues not less than three. Therefore $\lambda_3(G_7(p, q)) < 3$. \square

Lemma 2.7. Let $G_8(p, q)$ be the graph of order $n = 2p + q + 6$ in Fig. 5, where $p \geq 0$, $q \geq 0$. Then $\lambda_3(G_8(p, q)) < 3$.

Proof. By Lemma 2.1, we may assume that $q = 0$ and $p \geq 3$. By a direct calculation, we can show that the characteristic polynomial of $L(G_8(p, 0))$ is equal to

$$\begin{aligned} & \lambda(\lambda - 2)^2(\lambda^2 - 3\lambda + 1)^{p-1}(\lambda^5 - (p + 11)\lambda^4 + (9p + 42)\lambda^3 \\ & \quad - (25p + 65)\lambda^2 + (22p + 35)\lambda - (2p + 6)) \\ & := \lambda(\lambda - 2)^2(\lambda^2 - 3\lambda + 1)^{p-1}g(\lambda). \end{aligned}$$

It is easy to see that $g(0) < 0$, $g(1) > 0$, $g(2) < 0$, $g(3) > 0$, $g(5) < 0$, $g(p+4) > 0$. Hence $g(\lambda) = 0$ has exactly two roots not less than three. So $L(G_8(p, 0))$ has exactly two eigenvalues not less than three. Therefore $\lambda_3(G_8(p, q)) < 3$. \square

Lemma 2.8. Let $G_9(p, q, r, s)$ be the graph of order $n = 2p + q + 2r + s + 5$ in Fig. 6, where $p, r, q, s \geq 0$. If $(6p + 3q - 4)(6r + 3s - 4) < 4$ or $\max\{6p + 3q - 4, 6r + 3s - 4\} \leq 0$,

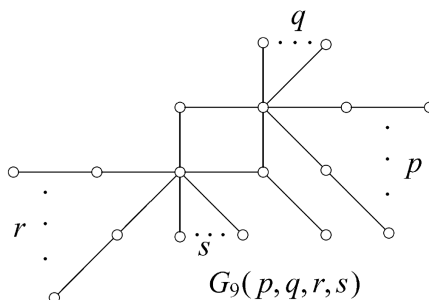


Fig. 6.

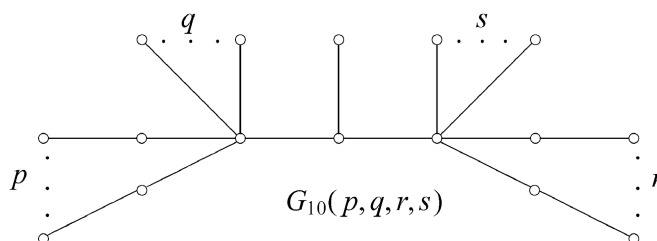


Fig. 7.

then $\lambda_3(G_9(p, q, r, s)) < 3$; if $(6p + 3q - 4)(6r + 3s - 4) \geq 4$ and $\max\{6p + 3q - 4, 6r + 3s - 4\} > 0$, then $\lambda_3(G_9(p, q, r, s)) \geq 3$.

Proof. If $p = q = 0$ or $r = s = 0$, then it follows from Lemma 2.3 that the assertion holds. If $p = 0, q = 1$; or $r = 0, s = 1$; then by Lemma 2.1, without loss of generality, we may assume that $r = 0, s = 1, q = 0, p \geq 1$. By a direct calculation, we can show that the characteristic polynomial of $L(G_9(p, 0, 0, 1))$ is equal to

$$\begin{aligned} & \lambda(\lambda^2 - 3\lambda + 1)^{p-1}(\lambda^7 - (p + 15)\lambda^6 + (12p + 89)\lambda^5 - (54p + 268)\lambda^4 \\ & + (115p + 435)\lambda^3 - (119p + 373)\lambda^2 + (54p + 155)\lambda - (8p + 24)) \\ & := \lambda(\lambda^2 - 3\lambda + 1)^{p-1}g(\lambda). \end{aligned}$$

It is easy to see that $g(0) < 0$, $g(\frac{1}{2}) > 0$, $g(1) < 0$, $g(2) < 0$, $g(\frac{8}{3}) < 0$, $g(3) > 0$, $g(5) < 0$, $g(p + 5) > 0$. Hence $g(\lambda) = 0$ has exactly two roots not less than three. So $L(G_9(p, 0, 0, 1))$ has exactly two eigenvalues not less than three. Therefore $\lambda_3(G_9(p, q, 0, 1)) < 3$.

If $p \geq r \geq 1$; or $r = 0, s = 2, p \geq 1$; or $r = 0, s = 2, p = 0, q \geq 2$; then by a similar argument, we may show $\lambda_3(G_9(p, q, r, s)) \geq 3$. Hence the assertion holds. \square

Lemma 2.9. Let $G_{10}(p, q, r, s)$ be the tree of order $n = 2p + q + 2r + s + 4$ in Fig. 7, where $p, r, q, s \geq 0$. If $(6p + 3q - 8)(6r + 3s - 8) < 16$ or $\max\{6p + 3q - 8, 6r + 3s - 8\} \leq 0$, then $\lambda_3(G_{10}(p, q, r, s)) < 3$; if $(6p + 3q - 8)(6r + 3s - 8) \geq 16$ and $\max\{6p + 3q - 8, 6r + 3s - 8\} > 0$, then $\lambda_3(G_{10}(p, q, r, s)) \geq 3$.

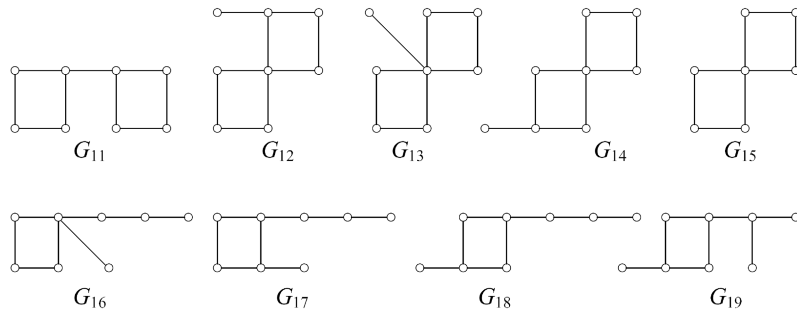


Fig. 8.

Proof. If $r \leq 1$, $s=0$; or $p=1$, $q=0$; then without loss of generality, we may assume that $r=1$, $s=0$, $q=p \geq 2$. By a direct calculation, we can show that the characteristic polynomial of $L(G_{10}(p, p, 1, 0))$ is equal to

$$\begin{aligned} & \lambda(\lambda-1)^{p-1}(\lambda^2-3\lambda+1)^{p-1}(\lambda^8-(2p+14)\lambda^7 \\ & + (24p+79)\lambda^6 - (111p+233)\lambda^5 + (251p+390)\lambda^4 \\ & - (291p+377)\lambda^3 + (167p+204)\lambda^2 - (42p+56)\lambda + (3p+6)) \\ & := \lambda(\lambda-1)^{p-1}(\lambda^2-3\lambda+1)^{p-1}g_1(\lambda). \end{aligned}$$

It is easy to see that $g_1(0) > 0$, $g_1(\frac{1}{3}) < 0$, $g_1(\frac{3}{5}) > 0$, $g_1(1) < 0$, $g_1(2) > 0$, $g_1(\frac{5}{2}) < 0$, $g_1(3) > 0$, $g_1(5) < 0$, $g_1(2p+3) > 0$. Hence $g_1(\lambda) = 0$ has exactly two roots not less than three. So $L(G_{10}(p, p, 1, 0))$ has exactly two eigenvalues not less than three. Therefore $\lambda_3(G_{10}(p, q, 1, 0)) < 3$.

For the other cases, by a similar argument, we may show the assertion holds and the details are omitted. \square

3. All connected bipartite graphs with $\lambda_3(G) < 3$

In this section, we characterize all connected bipartite graphs whose third largest Laplacian eigenvalue is less than 3. Denote by Γ_n the set of all connected bipartite graphs of order n which do not have any subgraphs isomorphic to one of the graphs $H_1 - H_8$ in Fig. 1 in Lemma 2.2. We firstly prove some Lemmas.

Lemma 3.1. *Let $G \in \Gamma_n$. If G contains a cycle of length 4, then G must be one of the graphs $G_1(p, q, r, s)$, $G_4(m, p, q, r, s)$, $G_5(p, q)$, $G_6(p, q)$, $G_7(p, q)$, $G_8(p, q)$, $G_9(p, q, r, s)$, G_{11}, \dots, G_{19} Fig. 8.*

Proof. Let the vertex set of a bipartite graph G be $V = \{v_1, \dots, v_n\}$. Clearly, G contains a cycle $v_1v_2v_3v_4v_1$ of length 4 as the induced subgraph. We consider the following two cases.

Case 1: The induced subgraph of G by vertex set $U = \{v_5, \dots, v_n\}$ contains a cycle $v_5v_6v_7v_8v_5$ of length 4. Since H_1 is a forbidden subgraph, there exists a vertex, say v_5 , in $\{v_5, v_6, v_7, v_8\}$ such that v_5 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_5 \sim v_1$, where \sim stands for “adjacency relationship”. We claim $n=8$. In fact, if there exists a vertex v_9 adjacent to v_i , for i with $1 \leq i \leq 8$, then G contains H_1 as a subgraph and it is a contradiction. Moreover, since H_2 and H_3 are forbidden subgraphs, it is easy to see that $v_i \not\sim v_j$ for $i=1, 2, 3, 4$; $j=6, 7, 8$ and $i=2, 3, 4$; $j=5$. G must be G_{11} .

Case 2: The induced subgraph $G[U]$ of G by vertex set $U = \{v_5, \dots, v_n\}$ does not contain a cycle of length 4. Then $G[U]$ does not contain any cycles and a path of order 6 either. If there exists a vertex, say v_5 , in U such that v_5 is adjacent to two vertices in $\{v_1, v_2, v_3, v_4\}$, say, $v_5 \sim v_1$ and $v_5 \sim v_3$, then v_i is not adjacent to v_2, v_4 and v_5 for $i=6, \dots, n$, since H_2 is a forbidden subgraph. Moreover, it is easy to see that G must be $G_4(m, p, q, r, s)$. Hence we now assume that any vertex in U is adjacent to at most a vertex in $\{v_1, v_2, v_3, v_4\}$ and consider the following five subcases:

Subcase 2.1: The induced subgraph $G[U]$ contains a path of order 5, say $v_i \sim v_{i+1}$ for $i=5, 6, 7, 8$. Since H_1 is a forbidden subgraph, it is easy to see that $v_i \not\sim v_j$ for $i=1, 2, 3, 4$; $j=5, 6, 8, 9$.

If v_7 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_7 \sim v_1$, then $v_i \not\sim v_3$ for $i=5, \dots, n$, since H_5 is a forbidden subgraph. If there exists a vertex in $\{v_{10}, \dots, v_n\}$, say v_{10} such that $v_{10} \sim v_1$, then $v_i \not\sim v_j$ for $i=11, \dots, n$; $j=1, \dots, 9$, $j \neq 7$ and G must be $G_7(p, q)$. If there exists a vertex, say v_{10} , in $\{v_{10}, \dots, v_n\}$ such that v_{10} is adjacent to v_2 or v_4 , then it is easy to see that G must be $G_5(p, q)$. If $v_i \not\sim v_j$ for $i=10, \dots, n$; $j=1, 2, 3, 4$, then G must be $G_6(p, q)$.

If v_7 is not adjacent to any vertex in $\{v_1, v_2, v_3, v_4\}$, then there exists a vertex, say v_{10} in $\{v_{10}, \dots, v_n\}$ such that $v_{10} \sim v_7$ and v_{10} is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_{10} \sim v_1$, since G is connected and H_1 is a forbidden subgraph. Hence G must be $G_8(p, q)$.

Subcase 2.2: The induced subgraph $G[U]$ does not contain a path of order 5 but contains a path of order 4, say $v_i \sim v_{i+1}$ for $i=5, 6, 7$. Moreover, without loss of generality, we may assume that there exists a vertex in $\{v_5, v_6, v_7, v_8\}$ such that it is adjacent to a vertex in $\{v_1, \dots, v_4\}$.

If v_5 or v_8 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_5 \sim v_1$, then $v_i \not\sim v_j$ for $i=5, 6, 7, 8$; $j=2, 3, 4$; and $v_i \not\sim v_1$ for $i=6, 8$. If $v_7 \sim v_1$, then $n=8$ and G must be G_{12} , since H_1 is a forbidden subgraph. If $v_7 \not\sim v_1$, then $v_i \not\sim v_j$ for $i=9, \dots, n$; $j=1, 2, 3, 4, 5, 7, 8$, since H_1 is a forbidden subgraph. Hence G must be $G_8(1, q)$.

If $v_i \not\sim v_j$ for $i=5, 8$; $j=1, 2, 3, 4$, then v_6 or v_7 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_6 \sim v_1$, which implies $v_7 \not\sim v_j$ for $j=1, 2, 3, 4$. Since H_5 is a forbidden subgraph, $v_i \not\sim v_3$ for $i=9, \dots, n$. Moreover, $v_i \not\sim v_j$ for $i=9, \dots, n$; $j=5, 7, 8$. If there exists a vertex, say v_9 in $\{v_9, \dots, v_n\}$ such that $v_9 \sim v_1$, then G must be $G_7(1, q)$. If there exists a vertex, say v_9 in $\{v_9, \dots, v_n\}$ such that $v_9 \sim v_2$ or v_4 , then G must be $G_5(1, q)$. If $v_i \not\sim v_j$ for $i=9, \dots, n$; $j=1, 2, 3, 4$, then G must be $G_6(1, q)$.

Subcase 2.3: The induced subgraph $G[U]$ does not contain a path of order 4 but contains a path of order 3, say $v_i \sim v_{i+1}$ for $i=5, 6$.

Moreover, we may assume that there exists a vertex in $\{v_5, v_6, v_7\}$ such that it is adjacent to a vertex in $\{v_1, \dots, v_4\}$, since G is a connected graph.

If v_5 or v_7 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_5 \sim v_1$, then $v_6 \not\sim v_j$ for $j = 1, 2, 3, 4$ and $v_7 \not\sim v_j$ for $j = 2, 3, 4$. Moreover, if $v_7 \sim v_1$, then it is easy to see that there exists at most an edge between $\{v_1, \dots, v_7\}$ and $\{v_8, \dots, v_n\}$, since H_1 is a forbidden subgraph. Hence $n \leq 8$ and G must be one of the graphs G_{12} , G_{13} , G_{14} and G_{15} . If $v_7 \not\sim v_1$ and there exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that v_8 is adjacent to a vertex v_t in $\{v_1, v_2, v_3, v_4\}$, $1 \leq t \leq 4$, then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, \dots, 8$, $j \neq t$, since H_1 and H_5 are forbidden subgraphs. Therefore $n = 8$ and G must be one of the graphs G_{16} , G_{17} and G_{18} , since H_1 are forbidden subgraphs. Hence we may assume that $v_7 \not\sim v_1$ and $v_i \not\sim v_j$ for $i = 8, \dots, n$; $j = 1, 2, 3, 4$. Then G must be $G_8(0, q)$.

If $v_i \not\sim v_j$ for $i = 5, 7$; $j = 1, 2, 3, 4$, then v_6 is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$, say $v_6 \sim v_1$. If there exists a vertex say, v_8 , in $\{v_8, \dots, v_n\}$, such that $v_8 \sim v_3$, then $n = 8$ and G must be G_{19} , since H_1 and H_6 are forbidden subgraphs. Hence we now assume that $v_i \not\sim v_3$ for $i = 8, \dots, n$. If there exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that $v_8 \sim v_1$, then G must be $G_7(0, q)$. If there exists a vertex, say v_8 in $\{v_8, \dots, v_n\}$ such that v_8 is adjacent to v_2 or v_4 , then G must be $G_5(0, q)$. If $v_i \not\sim v_j$ for $i = 8, \dots, n$; $j = 1, 2, 3, 4$, then G must be $G_6(0, q)$.

Subcase 2.4: The induced subgraph $G[U]$ does not contain a path of order 3 but contains at least one edge, say $v_5 \sim v_6$. Moreover, we assume that $v_5 \sim v_1$.

If there exists a vertex, say v_7 , in $\{v_7, \dots, v_n\}$ such that v_7 is adjacent to v_2 or v_4 , say $v_7 \sim v_2$, then $v_7 \not\sim v_j$ for $j = 1, 3, 4, 5, 6$. Since H_4 is a forbidden subgraph, $v_i \not\sim v_4$ for $i = 8, \dots, n$. If there exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that $v_8 \sim v_3$, then $v_i \not\sim v_j$ for $i = 8, \dots, n$; $j = 2, 4, 7$, since H_4 , H_7 and H_8 are forbidden subgraphs. Hence G must be $G_9(p, q, r, s)$. If $v_i \not\sim v_3$ for $i = 8, \dots, n$, then G must be $G_1(p, q, r, s)$.

If $v_i \not\sim v_j$ for $i = 7, \dots, n$; $j = 2, 4$, then G must be $G_4(2, p, q, r, s)$.

Subcase 2.5: The induced subgraph $G[U]$ consists of isolated vertices.

Since H_8 is a forbidden subgraph, then there exists a vertex in $\{v_1, v_2, v_3, v_4\}$, say v_4 such that v_4 is not adjacent to v_i for $i = 5, \dots, n$. If $v_i \not\sim v_2$ for $i = 5, \dots, n$, then G must be $G_4(2, 0, q, 0, s)$. If there exists only a vertex, say v_5 , in $\{v_5, \dots, v_n\}$ such that $v_5 \sim v_2$, then G must be $G_9(0, q, 0, s)$. Hence we may assume that there exist at least two vertices, say v_5, v_6 , in $\{v_5, \dots, v_n\}$ such that $v_5 \sim v_2$ and $v_6 \sim v_2$. If there exists a vertex, say v_7 , in $\{v_7, \dots, v_n\}$ such that $v_7 \sim v_1$, then $v_i \not\sim v_3$ for $i = 8, \dots, n$, since H_7 is a forbidden subgraph. Hence G must be $G_1(0, q, 0, s)$. If $v_i \not\sim v_1$ for $i = 7, \dots, n$, then G must be $G_1(0, q, 0, s)$. \square

Lemma 3.2. Let $G \in \Gamma_n$. If G is a tree, then G must be one of the graphs $G_2(p, q, r, s)$, $G_3(p, q, r, s)$ $G_{10}(p, q, r, s)$.

Proof. Let $V = \{v_1, \dots, v_n\}$ be the vertex set of G . Since H_1 is a forbidden subgraph, G does not contain a path of order 9 as a subgraph. Hence we consider the following five cases:

Case 1: G contains a path of order 8, say $v_i \sim v_{i+1}$ for $i = 1, \dots, 7$.

Since G is tree, the path $v_1 \dots v_8$ is the induced subgraph of G and $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 2, 4, 5, 7, 8$, since H_1 is a forbidden subgraph. On the other hand, since H_1 is a forbidden subgraph of G , then the induced subgraph of G by the vertex

set $U = \{v_9, \dots, v_n\}$ consists of separated edges and isolated vertices. Hence G must be $G_2(p, q, r, s)$.

Case 2: G does not contain a path of order 8 but contains a path of order 7, say $v_i \sim v_{i+1}$ for $i = 1, \dots, 6$. Since G is a tree, the path $v_1 \dots v_7$ is the induced subgraph of G . We consider the following two subcases:

Subcase 2.1: There exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that $v_8 \sim v_4$. Then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 2, 4, 6, 7, 8$, since H_1 is a forbidden subgraph. On the other hand, the induced subgraph $G[v_9, \dots, v_n]$ consists of separated edges and isolated vertices. Hence G must be $G_{10}(p, q, r, s)$.

Subcase 2.2: $v_i \not\sim v_4$ for $i = 8, \dots, n$. If there exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that v_8 is adjacent to v_2 or v_6 , say $v_8 \sim v_2$. Then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 3, 4, 6, 7, 8$. On the other hand, The induced subgraph $G[v_9, \dots, v_n]$ consists of separated edges and isolated vertices. Hence G must be $G_2(p, q, 0, s)$. If $v_i \not\sim v_j$ for $i = 8, \dots, n$; $j = 2, 4, 6$, then G must be $G_4(1, p, q, r, s)$, since the induced subgraph $G[v_8, \dots, v_n]$ consists of separated edges and isolated vertices.

Case 3: G does not contain a path of order 7 but contains a path of order 6, say $v_i \sim v_{i+1}$ for $i = 1, \dots, 5$.

Since G is a tree, the path $v_1 \dots v_6$ is the induced subgraph of G . We consider the following three subcases:

Subcase 3.1: There exist two vertices, say v_7, v_8 , in $\{v_7, \dots, v_n\}$ such that $v_7 \sim v_2$, $v_8 \sim v_5$. Then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 3, 4, 6$. Hence G must be $G_2(0, q, 0, s)$.

Subcase 3.2: There exists a vertex, say v_7 in $\{v_7, \dots, v_n\}$ such that $v_7 \sim v_2$ and $v_i \not\sim v_5$ for $i = 8, \dots, n$. Moreover, if there exists a vertex, say v_8 , in $\{v_8, \dots, v_n\}$ such that $v_8 \sim v_3$, then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 3, 5, 6, 7, 8$. On the other hand the induced subgraph $G[v_7, \dots, v_n]$ consists of separated edges and isolated vertices. Hence G must be $G_{10}(p, q, 0, s)$. If $v_i \not\sim v_3$ for $i = 8, \dots, n$, then G must be $G_4(1, p, q, 0, s)$.

Subcase 3.3: $v_i \not\sim v_j$ for $i=7, \dots, n$; $j=2, 5$. Clearly, the induced subgraph $G[v_7, \dots, v_n]$ consists of separated edges and isolated vertices. Then G must be $G_3(p, q, r, s)$.

Case 4: G does not contain a path of order 6 but contains a path of order 5, say $v_i \sim v_{i+1}$ for $i = 1, \dots, 4$.

Since G is tree, the path $v_1 \dots v_5$ is the induced subgraph of G . We consider the following three subcases:

Subcase 4.1: $v_i \not\sim v_3$ for $i = 6, \dots, n$. Then it is easy to see that G must be $G_4(1, 0, q, 0, s)$.

Subcase 4.2: There exist three vertices, say v_6, v_7, v_8 , in $\{v_6, \dots, v_n\}$ such that $v_6 \sim v_3$, $v_7 \sim v_2$, $v_8 \sim v_4$.

Then $v_i \not\sim v_j$ for $i = 9, \dots, n$; $j = 1, 3, 5, 6, 7, 8$ since G does not contain a path of order 6 and H_1 is a forbidden subgraph. Hence G must be $G_{10}(0, q, 0, s)$.

Subcase 4.3: $v_i \sim v_3$ and either $v_i \not\sim v_2$ or $v_i \not\sim v_4$ for $i = 6, \dots, n$.

We may assume that $v_i \not\sim v_4$ for $i = 6, \dots, n$. It is easy to see that G must be $G_3(p, q, 0, s)$.

Case 5: G does not contain a path of order 5. It is easy to see that G must be $G_3(0, q, 0, s)$. \square

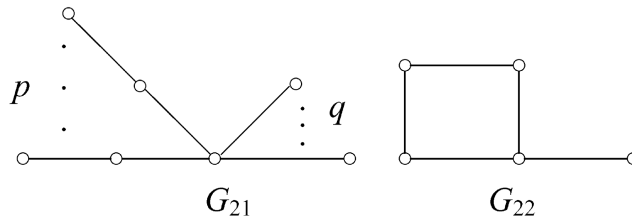


Fig. 9.

We are ready to present the main result of this paper.

Theorem 3.3. *A connected bipartite graph G of order n belongs to \mathcal{G} , i.e., $\lambda_3(G) < 3$ if and only if G is one of the following graphs: $G_1(p, q, r, s)$, $G_2(p, q, r, s)$, $G_3(p, q, r, s)$, $G_4(m, p, q, r, s)$, $G_5(p, q)$, $G_6(p, q)$, $G_7(p, q)$, $G_8(p, q)$, $G_9(p, q, r, s)$ with $(6p + 3q - 4)(6r + 3s - 4) < 4$ or $\max\{6p + 3q - 4, 6r + 3s - 4\} \leq 0$, $G_{10}(p, q, r, s)$ with $(6p + 3q - 8)(6r + 3s - 8) < 16$ or $\max\{6p + 3q - 8, 6r + 3s - 8\} \leq 0$, G_{11}, \dots, G_{19} .*

Proof. It directly follows from Lemmas 3.1, 3.2, 2.5, 2.6, 2.7, 2.8 and 2.9, since the cycle of order 8 is a forbidden subgraph. \square

4. Bipartite graphs characterized by $\lambda_2(G) \geq 3$ and $\lambda_3(G) < 3$

In order to characterize all connected graphs which have exactly two Laplacian eigenvalues not less than three, we need the following two Lemmas:

Lemma 4.1. *Let $G_{21}(p, q)$ and G_{22} be connected graphs of order $n = 2p + q + 1$ and order 5, respectively in Fig. 9, where $p, q \geq 0$. Then $\lambda_2(G_{21}(p, q)) < 3$ and $\lambda_2(G_{22}) < 3$.*

Proof. By a direct calculation, $\lambda_2(G_{22}) < 3$. By Lemma 2.1, without loss of generality, we may assume that $q = 0$, $p \geq 1$, since $G_{21}(p, q)$ can be regarded as the subgraph of $G_{21}(p + q, p + q)$. By a direct calculation, the characteristic polynomial of $L(G_{21}(p, 0))$ is equal to

$$\lambda(\lambda^2 - 3\lambda + 1)^{p-1}(\lambda^2 - (p+3)\lambda + (2p+1)).$$

Hence $\lambda_2(G_{21}(p, q)) < 3$. \square

Lemma 4.2. *Let G be a connected graph of order n . Then $\lambda_2(G) < 3$ if and only if G is one of $G_{21}(p, q)$, G_{22} and the cycle of length 4.*

Proof. If G is G_{21} , G_{22} or a cycle of length 4, then by Lemma 4.1, we have $\lambda_2(G) < 3$.

Conversely, we assume that $\lambda_2(G) < 3$ and vertex set $V = \{v_1, \dots, v_n\}$. By Lemma 2.1, it is easy to see that G does not contain a cycle of order 3, or $k \geq 5$. If G contains

a cycle $v_1v_2v_3v_4v_1$ of length 4, then it is easy to see that there is at most one vertex in $\{v_5, \dots, v_n\}$ such that it is adjacent to a vertex in $\{v_1, v_2, v_3, v_4\}$. Hence G must be G_{22} or a cycle of order 4. Now we may assume that G does not contain a cycle of order $k \geq 3$. Then G is tree. Clearly, G does not contain a path of order 6. So we consider the following three cases:

Case 1: G contains a path of order 5, say $v_i \sim v_{i+1}$ for $i = 1, 2, 3, 4$. Then $v_i \not\sim v_j$ for $i = 6, \dots, n$; $j = 1, 2, 4, 5$. Hence G must be $G_{21}(p, q)$.

Case 2: G does not contain a path of order 5 but contains a path of order 4, say $v_i \sim v_{i+1}$ for $i = 1, 2, 3$. Clearly, there do not exist two vertices, say u, v in $\{v_5, \dots, v_n\}$ such that $u \sim v_2$ and $v \sim v_3$. Hence G must be $G_{21}(1, q)$.

Case 3: G does not contain a path of order 4. Then it is easy to see that G must be $G_{21}(0, q)$. \square

We present the main result in this section.

Theorem 4.3. *Let G be a connected bipartite graph of order n . Then G has exactly two Laplacian eigenvalues not less than three if and only if G is one of the following graphs: $G_1(p, q, r, s)$, $G_2(p, q, r, s)$, $G_3(p, q, r, s)$, $G_4(m, p, q, r, s)$, $G_5(p, q)$, $G_6(p, q)$, $G_7(p, q)$, $G_8(p, q)$, $G_9(p, q, r, s)$ with $(6p + 3q - 4)(6r + 3s - 4) < 4$ or $\max\{6p + 3q - 4, 6r + 3s - 4\} \leq 0$, $G_{10}(p, q, r, s)$ with $(6p + 3q - 8)(6r + 3s - 8) < 16$ or $\max\{6p + 3q - 8, 6r + 3s - 8\} \leq 0$, G_{11}, \dots, G_{19} , but G is not one of graphs $G_{21}(p, q)$, G_{22} and the cycle of length 4.*

Proof. It follows from Theorem 3.3 and Lemma 4.2. \square

5. Line graphs of bipartite graphs

In this section, we present an application of our main result. Let $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ be the eigenvalues of the $(0, 1)$ adjacency matrix of a graph G . Several researcher studied the graphs with property $\mu_2(G) < 1$. In particular, Petrovic [12] characterized all bipartite graphs with $\mu_2(G) \leq 1$. Cao and Hong [3] characterized all graphs with $0 < \mu_2(G) < \frac{1}{3}$. Petrovic and Milekic [14] characterized all line graphs with $\mu_2(G) \leq 1$. Here, all line graphs G^l of bipartite graphs G with $\mu_3(G^l) < 1$ are characterized. The next Lemma is well known, see, e.g., [13].

Lemma 5.1. *Let G be a bipartite graph of order n and G^l be its line graph. Then $\lambda_i(G) = 2 + \mu_i(G^l)$, for $i = 1, \dots, n - 1$.*

Theorem 5.2. *Let G be a bipartite graph and G^l be its line graph. Then $\mu_3(G^l) < 1$ if and only if G^l is the line graph of one of the following graphs $G_1(p, q, r, s)$, $G_2(p, q, r, s)$, $G_3(p, q, r, s)$, $G_4(p, q, r, s)$, $G_5(m, p, q, r, s)$, $G_6(p, q)$, $G_7(p, q)$, $G_8(p, q)$, $G_9(p, q, r, s)$ with $(6p + 3q - 4)(6r + 3s - 4) < 4$ or $\max\{6p + 3q - 4, 6r + 3s - 4\} \leq 0$, $G_{10}(p, q, r, s)$ with $(6p + 3q - 8)(6r + 3s - 8) < 16$ or $\max\{6p + 3q - 8, 6r + 3s - 8\} \leq 0$, G_{11}, \dots, G_{19} .*

Proof. It follows from Theorem 3.3 and Lemma 5.1. \square

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